The break minimization problem is solvable in polynomial time when the optimal value is less than the number of teams*

Ryuhei Miyashiro and Tomomi Matsui

Department of Mathematical Informatics,
Graduate School of Information Science and Technology,
The University of Tokyo, Tokyo 113-0033, Japan
{miya, tomomi}@misojiro.t.u-tokyo.ac.jp

Abstract. The break minimization problem is to find a home-away assignment that minimizes the number of breaks for a given timetable of a round-robin tournament. In a recent paper, Elf, Jünger and Rinaldi conjectured that the break minimization problem is solvable in polynomial time when the optimal value is less than the number of teams. This paper proves their conjecture affirmatively by showing that a problem related to the break minimization problem is solvable in polynomial time. Our approach is to transform an instance of the related problem into instances of the 2-satisfiability problem.

In this paper, we prove a previously proposed conjecture about the break minimization problem in sports timetabling.

We consider a round-robin tournament with the following properties:

- the number of teams is $2n$ ($n \in \mathbb{N}$), and the number of *slots*, i.e., the days when matches are held, is $2n-1$;
- each team plays one match in each slot;
- each team plays every other team once;
- each team has its home, and each match is held at the home of one of the corresponding two teams.

Figure 1 is a *timetable* of a round-robin tournament satisfying these properties. In the figure, each match with '@' means that the match is held at the home of the opponent; without '@' is held at the home of the team corresponding to the row. For example, in slot 5 team 2 plays team 3 at the home of team 2. In other words, team 3 plays *at away* in slot 5, whereas team 2 plays *at home*.

If a team plays either both at home or both at away in slots $s-1$ and s, it is said that the team has a *break* at slot s. In Fig. 1, team 3 plays at home in slots 1 and 2, and thus team 3 has a break at slot 2. In total, the timetable has six breaks, each of which is represented as a line under the corresponding entry.

Given a timetable not assigned where to play, one should decide a *home-away assignment* to complete a timetable (Fig. 2). In most of practical sports

* Supported by Superrobust Computation Project of the 21st Century COE Program "Information Science and Technology Strategic Core," Japan.

	1	2	3	4	5 (slot)
1 :	@6	@$\underline{3}$	@$\underline{5}$	2	@4
2 :	@5	6	$\underline{4}$	@1	3
3 :	4	$\underline{1}$	@6	5	@2
4 :	@3	5	@2	6	$\underline{1}$
5 :	2	@4	1	@3	@$\underline{6}$
6 :	1	@2	3	@4	5
(team)					

Fig. 1. A timetable with six breaks

timetabling, a break is considered undesirable and the number of breaks in a timetable is required to be reduced. In this context, the *break minimization problem* is introduced as follows.

Break Minimization Problem

Instance: A timetable without a home-away assignment.

Task: Find a home-away assignment that minimizes the number of breaks for a given instance.

The timetable without a home-away assignment of Fig. 2 is an instance of the break minimization problem. Although the timetable of Fig. 1 shows a feasible home-away assignment for the instance, it is suboptimal. The timetable of Fig. 2 is an optimal solution, whose optimal value is four.

There are some previous results on the break minimization problem: Régin [6] solved up to 20 teams instances with constraint programming; Trick [7] proposed integer programming formulations and solved instances up to 22 teams; Elf, Jünger and Rinaldi [3] formulated this problem as MAX CUT, and solved instances up to 26 teams. All of them are exact methods based on branch-and-bound techniques. The authors [5] formulated this problem as MAX RES CUT and MAX 2SAT, and solved instances up to 40 teams with Goemans and Williamson's approximation algorithm for MAX RES CUT [4], though the obtained solutions are not necessarily optimal.

There are some open problems about break minimization. Although it is conjectured that the break minimization problem is NP-hard, the complexity status is not yet determined. Concerning the complexity, Elf et al. [3] reported the following results: their instances of the break minimization problem were solved very quickly when the instances had the optimal value $2n-2$. (The value $2n-2$ is a lower bound of the objective value for any instance of $2n$ teams, because a timetable of $2n$ teams has at least $2n-2$ breaks [2].) According to their experience, they conjectured that the break minimization problem is solvable in polynomial time if a given instance of $2n$ teams has the optimal value $2n-2$.

We prove their conjecture affirmatively by showing that the following problem is solvable in polynomial time.

	1	2	3	4	5
1 :	6	3	5	2	4
2 :	5	6	4	1	3
3 :	4	1	6	5	2
4 :	3	5	2	6	1
5 :	2	4	1	3	6
6 :	1	2	3	4	5

	1	2	3	4	5
1 :	@6	3	@5	2	@4
2 :	@5	6	$\underline{4}$	@1	3
3 :	4	@1	@$\underline{6}$	5	@2
4 :	@3	5	@2	6	$\underline{1}$
5 :	2	@4	1	@3	@$\underline{6}$
6 :	1	@2	3	@4	5

Fig. 2. A timetable without a home-away assignment and with an optimal assignment

Problem (P)

Instance: A timetable of $2n$ teams without a home-away assignment.

Task: Find a home-away assignment with $2n-2$ breaks for a given instance, or decide that none exists.

In the following, we show that Problem (P) is solvable in $O(n^3)$ steps. For Problem (P), we define Subproblem (P_k) as follows ($k \in T$, where T is a set of teams, i.e., $\{1, 2, \ldots, 2n\}$). It is not difficult to see that Problem (P) is feasible if and only if at least one of (P_1), (P_2), ..., and (P_{2n}) is feasible.

Subproblem (P_k)

Task: For a given instance of Problem (P), find a home-away assignment with $2n-2$ breaks in which team k has no breaks and plays at home in slot 1, or decide that none exists.

The feasibility of Subproblem (P_k) is equivalent to that of Subproblem (P'_k) defined below. In addition, a feasible home-away assignment of (P_k) can be constructed from that of (P'_k), by substituting home/away for away/home in all even slots. (More generally, the following statement holds: by the above substitution, for a given instance an optimal solution of the break minimization problem is obtained from that of the *break maximization problem* and vice versa.)

Subproblem (P'_k)

Task: For a given instance of Problem (P), find a home-away assignment with $2n(2n-2) - (2n-2)$ breaks in which team k has $2n-2$ breaks and plays at home in slot 1, or decide that none exists. (In other words, team k plays only at home and every other team has at most one "non-break.")

Now we formulate Subproblem (P'_k) ($k \in T$) as the 2-satisfiability problem (2SAT). Let S be a set of slots, i.e., $\{1, 2, \ldots, 2n-1\}$. We define a Boolean variable $x_{t,s}$ ($t \in T$, $s \in S$) as follows: a variable $x_{t,s}$ is `FALSE` if team t plays at home in slot s, otherwise `TRUE`. Then, an instance of Subproblem (P'_k) is described as follows.

$$
\begin{array}{lll}
\text{Find} & x_{t,s} \in \{\texttt{TRUE}, \texttt{FALSE}\} & (\forall t \in T,\ \forall s \in S) \\
\text{s. t.} & x_{k,s} = \texttt{FALSE} & (\forall s \in S), \\
 & x_{t,s} \neq x_{\tau(t,s),s} & (\forall t \in T,\ \forall s \in S), \\
 & \neg x_{t,s} \vee x_{t,s+1} & (\forall t \in T \setminus \{k\},\ \forall s \in S,\ s < s_{k,t}), \\
 & x_{t,s-1} \vee \neg x_{t,s} & (\forall t \in T \setminus \{k\},\ \forall s \in S,\ s > s_{k,t}), \\
 & x_{t,1} \vee x_{t,2n-1} & (\forall t \in T \setminus \{k\})
\end{array}
$$

where $\tau(t, s)$: the opponent of team t in slot s in the given instance;
$s_{k,t}$: the slot when team k plays team t in the given instance.

Each constraint can be represented as clause(s) with two literals. Thus, this is an instance of 2SAT; both the number of variables and that of clauses are $O(n^2)$. Since 2SAT with p literals and q clauses can be solved in $O(p + q)$ steps [1], Problems (P'_k) and (P) are solvable in $O(n^2)$ and $O(n^3)$ steps, respectively.

Finally, we mention a home-away assignment with $2n$ breaks. A home-away assignment in which each team has exactly one break is said to be *equitable* [2], and it is sometimes preferred to a home-away assignment with $2n - 2$ breaks. We note that the following problem is also solvable in $O(n^3)$ steps.

Problem ($\bar{P}$)
Instance: A timetable of $2n$ teams without a home-away assignment.
Task: Find a home-away assignment in which each team has exactly one break for a given instance, or decide that none exists.

References

1. Aspval, B., Plass, M.F., Tarjan, R.E.: A Linear-Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas. Information Processing Letters **8** (1979) 121–123
2. de Werra, D.: Geography, Games and Graphs. Discrete Applied Mathematics **2** (1980) 327–337
3. Elf, M., Jünger, M., Rinaldi, G.: Minimizing Breaks by Maximizing Cuts. Operations Research Letters **31** (2003) 343–349
4. Goemans, M.X., Williamson, D.P.: Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming. Journal of the ACM **42** (1995) 1115–1145
5. Miyashiro, R., Matsui, T.: Semidefinite Programming Based Approaches to the Break Minimization Problem. Mathematical Engineering Technical Reports, Department of Mathematical Informatics, Graduate School of Information Science and Technology, the University of Tokyo, METR 2003-28 (2003) (*available at* http://www.keisu.t.u-tokyo.ac.jp/Research/techrep.0.html)
6. Régin, J.-C.: Minimization of the Number of Breaks in Sports Scheduling Problems Using Constraint Programming. In: Freuder, E.C., Wallace, R.J. (eds.): Constraint Programming and Large Scale Discrete Optimization. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 57. American Mathematical Society, Providence (2001) 115–130
7. Trick, M.A.: A Schedule-Then-Break Approach to Sports Timetabling. In: Burke, E., Erben, W. (eds.): Practice and Theory of Automated Timetabling III (PATAT 2000, Konstanz, Germany, August, selected papers). Lecture Notes in Computer Science, Vol. 2079. Springer-Verlag, Berlin Heidelberg New York (2001) 242–253