Partially Concurrent Open Shop Scheduling with Preemption and Limited Resources

Hagai Ilani · Tal Grinshpoun · Elad Shufan

Abstract Partially Concurrent Open Shop Scheduling (PCOSS) is a relaxation of the well-known Open Shop Scheduling (OSS) problem, where some of the operations that refer to the same job may be processed concurrently. Here we extend the study of the PCOSS model by considering the addition of limited resources. We deal with the case of preemption PCOSS, where a few polynomial algorithms are known for its OSS counterpart. The scheduling problem is equivalent to the problem of conflict graph colouring. The restriction on the number of resources bounds the size of colour classes. We thus study the problem of bounded vertex colouring and focus on bounds. In particular, we introduce a new bound for this problem, propose a colouring procedure that is inspired by this new bound, and show that for perfect graphs with two resources the procedure attains the bound, and hence is optimal. The model correlates to a real-life timetabling project of assigning technicians to vehicles in a garage, with additional resources, such as vehicle lifts.

Keywords Graph colouring · Open shop scheduling · Concurrent machines · Limited resources · Technician timetabling

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1 Introduction

The *partially concurrent open shop scheduling* (PCOSS) problem deals with jobs that should be processed by machines [4,5]. It was motivated by a timetabling problem of assigning technicians (machines) to vehicles (jobs) in a garage. Similarly to the open shop problem, each machine processes at most one job at a time. But, for a given job, some of its operations may be processed concurrently and some may not. A conflict graph represents whether pairs of operations can or cannot be processed concurrently.

Many studies link between scheduling problems and graph colouring problems [12,1,13,10]. In our previous work, we have studied the problem of PCOSS with preemptions by its relation to the problem of conflict graph colouring [7,8]. For the Unit-Time processing scenario, the problem of minimising the makespan is equivalent to the classical problem of colouring the conflict graph with fewest colours. Similarly, for the integral-preemption scenario and general preemption scenario, minimising the makespan reduces to w-colouring and fractional w-colouring, respectively, of the weighted conflict graph with the processing times as its weights. Here, we assume the same problems with an additional resource for which we have r units at any given time. The reduction to graph colouring works the same as in the regular PCOSS with the additional restriction that each colour appears at most in r vertices, a problem known as the Bounded Colouring problem [6].[1]

It seems that the Bounded Vertex Colouring problem has so far yet to receive much research attention. Conversely, a related problem called the Equitable Graph Colouring Problem (EGC) has been investigated intensively [3]. In the EGC, a graph has to be (fractionally) (w-) coloured so that the sizes of all colour sets are as equal as possible. In terms of PCOSS, equitable colouring means that at any time the same number of resource units are busy.

The main contributions of this research are

1. Introducing a new bound for the r-bounded (fractional)-chromatic number (Section 2).
2. Defining a procedure for r-bounded colouring that combines regular colouring and equitable colouring (Section 3.1).
3. Proving that the above mentioned procedure attains the new bound for perfect graphs with $r = 2$ (Section 3.2).
4. Presenting an example with $r = 3$ for which the bound is not tight (Section 3.3).

2 PCOSS with resources

In a PCOSS problem a set of n jobs and m machines are given. The part of job j that has to be processed on machine k is called operation (j, k),

[1] The maximal number of instances that a colour should appear in the colouring is usually denoted as k. In the respective PCOSS problem, it is equivalent to the number of available resources. We will therefore refer to this number as r.

with $p_{j,k}$ denoting its processing time. In a standard Open Shop Scheduling (OSS) problem, any two operations that belong to the same machine, or to the same job, are not allowed to be processed concurrently. The PCOSS model generalizes this OSS model by allowing *some* operations (usually of the same job) to be processed concurrently. The set of operations that are not allowed to be processed concurrently are described by a conflict graph.

Definition 1 The *conflict graph* is a graph $G = (V, E)$. Each vertex $(j, k) \in V$ represents an operation,[2] with $j = 1, \dots, n$ and $k = 1, \dots, m$. Vertices (j, k) and (i, l) are adjacent if they may not be processed concurrently.

A PCOSS problem is defined by a set of n jobs, m machines, a processing time matrix $PT = [p_{j,k}]$, and a conflict graph G. The PCOSS problem that we study in this article is a *limited-resource* PCOSS. By this, we mean that each operation needs for its processing one unit of an additional resource. Each resource unit can be used by at most one operation at a time. The motivation comes from a vehicle garage, with jobs corresponding to vehicles and machines to technicians. The possible resource is a garage vehicle lift.

The usual objective considered hereby is the minimisation of the makespan C_{max}. The makespan is the time required to complete the last job, i.e., $C_{max} = max\{C_j \mid 1 \leq j \leq n\}$, where C_j is the completion time of job j.

There are two natural bounds for this PCOSS problem:

1. The clique bound: Each clique in the graph represents a set of operations that cannot be processed concurrently. Therefore, in the unit-time problem, the clique size bounds the makespan. In the case of general processing times, the bound is the total weight of the clique. We will denote this bound by B_c.
2. The resources bound: A resource is needed for any operation. Therefore, the makepan cannot be less than the total processing time divided by the number of resources, r. This bound is denoted by B_r.

Consider an example with $n = 3$ (jobs) and $m = 5$ (machines), a unit time processing time matrix

$$PT = \begin{pmatrix} 1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1 \end{pmatrix}, \tag{1}$$

and a conflict graph shown in Figure 1. In this example, $B_c = 3$ and $B_r = 15/r$.

In the next section, we will explicitly show that for this specific example and $r = 2$ or 5 the bounds are achievable. For a general OSS problem, it is known that when preemption is allowed, the makespan of an optimal schedule is always $max\{B_c, B_r\}$, with a polynomial algorithm that achieves this bound [2,13]. Are these bounds achievable for a general case of PCOSS? The following example shows that the answer is negative. Let us look at a unit-time PCOSS problem with $r = 2$ resources, one job, and a star-shaped conflict graph. In Figure 2(a), we show such a conflict graph with 5 machines. For this problem,

[2] Vertices that represent operations with zero processing times are omitted.

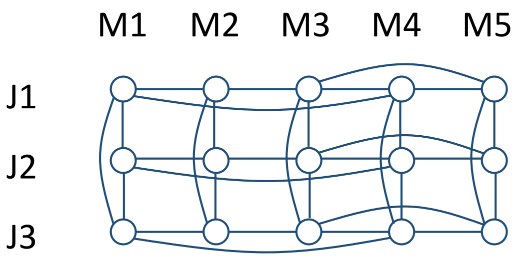

Fig. 1 An example of a PCOSS conflict graph.

$B_c = 2$ and $B_r = 5/2 = 2.5$. But, the makespan is larger: when the centered vertex is processed, only one resource can be used. This operation requires one unit of time, and the other four operations need two $(4/r)$ more additional time units. Inspired by this example, we suggest a new bound, called a mixed bound, denoted B_m. This bound is relevant when the vertices of $G = (V, E)$ can be divided into two subsets with the following characteristic: the induced graph $G_1 = (V_1, E_1)$ has a maximal clique of size ω_1, and all the n_2 vertices in subset $V_2 = V \setminus V_1$ are connected to all the vertices in the maximal clique in G_1. Figure 2(b) illustrates such a structure. Clearly, for a unit-time problem the operations of V_1 must use at least ω_1 time units. In addition, the operations related to V_2 must be processed in an additional n_2/r time units. The mixed bound is given by

$$B_m = \omega_1 + n_2/r. \tag{2}$$

Note that when $G_1 = \emptyset$ then B_m reduces to B_r, and when $G_2 = \emptyset$ it reduces to B_c. If the processing times are general, then ω_1 in Equation 2 denotes the weight of the maximum clique in G_1, and n_2 is the total weight of all V_2 vertices. For the star-shaped graph $B_m = 3$, which is achievable. The question is, what are the instances for which this bound is (polynomially) achievable? In the next section, we will discuss this question by the relation of the PCOSS problem with that of bounded graph colouring.

3 PCOSS with limited resources and bounded colouring

The relation between graph colouring and PCOSS scheduling has already been established [7,8]. A solution to a PCOSS scheduling problem is equivalent to the problem of colouring the conflict graph, with each colour corresponding to a unit of time. Vertices (operations) that can be processed simultaneously are not connected in the conflict graph, and can therefore share the same colour. In contrast, adjacent vertices must have different colours. Minimising the makespan in the scheduling problem is equivalent to minimising the number of colours needed for colouring the conflict graph, i.e., the makespan is equal to the chromatic number. In case there is an additional limited resource,

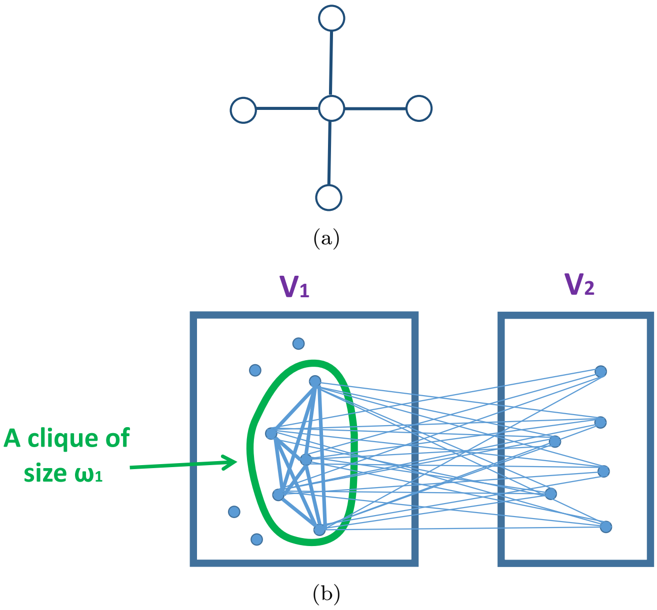

Fig. 2 (a) The resource bound and the clique bound are not achievable for the star shape conflict graph. (b) A conflict graph structure that defines the mixed bound.

the colouring is bounded, i.e., the size of each *colour class*[3] cannot be larger than r. Such colouring is called *bounded* vertex colouring [6]. Before giving more formal definitions, we shall return to the example of Figure 1. **Case 1.** Consider $r = 5$. In this case, the clique bound, $B_c = 3$, and the resource bound, $B_r = 15/5 = 3$, are equal. These bounds are achievable as shown by the proper colouring on the graph by three colours (Figure 3(a)). When the resource bound is achievable, the optimal schedule corresponds to an equitable colouring of the conflict graph. Here, each class of colours is exactly of size 5. **Case 2.** Consider $r = 2$. When only two resources are available a colour cannot appear in more than two vertices. The resource bound $B_r = 15/2 = 7.5$ is achievable. A possible *fractional* colouring that attains this bound is shown in Figure 3(b). Note that a vertex coloured by a fraction of a colour corresponds to an operation performed in the respective fraction of a time unit.

In this section, we will utilize the equivalence between the PCOSS scheduling and graph colouring problems by studying some aspects of bounded graph colouring problems and drawing conclusions regarding the achievable makespan bounds in the respective PCOSS with limited resources problems.

The following definitions are based on the terms of Hansen et al. [6].

Definition 2 Given a graph G and a number r, an r-bounded vertex colouring of G is a usual vertex colouring in which each colour is used at most r times.

[3] A colour class is the set of all vertices that share the same colour, which corresponds to operations that are processed concurrently.

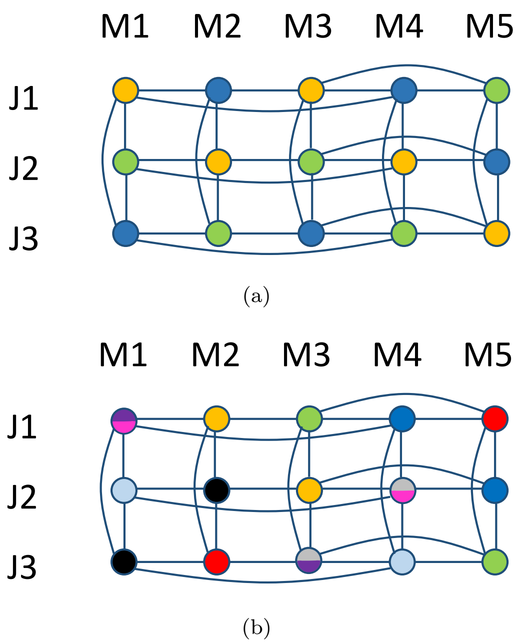

Fig. 3 Colouring of the conflict graph leads to a solution for a unit-time PCOSS with (a) $r = 5$ ($C_{max} = 3$). (b) $r = 2$ ($C_{max} = 7.5$).

Definition 3 The bounded chromatic number $\chi_r(G)$ of a graph G is the smallest number of colours such that G admits an r-bounded vertex colouring.

The above definitions correspond to a PCOSS with limited resources in which all processing times are equal (unit time) and no preemption is allowed. These definitions can be easily adapted to deal with non-unit processing times with preemption (either integral or general). For the case of integral preemption (with integral positive processing times), the graph $G = (V, E)$ in the above definitions is replaced with an integral-weighted graph (with $w(v)$ denoting the weight of vertex $v \in V$), and the colouring is replaced with a w-*proper* colouring,[4] in which $w(v)$ distinct colours are assigned to each vertex $v \in V$ so that adjacent vertices have no assigned colours in common. The w-proper colouring and its connection to PCOSS with integral preemption (not in the context of limited resources) are discussed in detail by Ilani et al. [7,8]. For the case of general preemption, the colouring is replaced with a fractional colouring when unit processing times are used, or with a fractional w-proper colouring for non-unit processing times. The fractional bounded chromatic number of a graph G with a limit of r resources is denoted $\chi^*_r(G)$.

We proceed to present a procedure for r-bounded colouring of a graph, which is at the focus of this section. We will then state theoretic properties of

[4] Also termed set colouring or multi-colouring in the graph colouring literature.

this procedure. On the one hand, we will show that when $r = 2$ the procedure attains the mixed bound B_m of Section 2 for perfect graphs. On the other hand, we will give a counter example for $r = 3$ in which the mixed bound is unachievable. For didactic reasons, in what follows we will focus on fractional (non-weighted) colouring (for general preemption with unit processing times), but these results can also be adjusted to deal with fractional w-proper colouring (for general preemption with non-unit processing times). Specifically, the r-bounded colouring procedure of the next subsection can also be adjusted to deal with non-fractional colouring (for integral preemption).

3.1 The r-bounded colouring procedure

The procedure starts by colouring the graph G with $\chi(G)$ colours. This colouring is named c. At the next stage of the procedure, the graph $G = (V, E)$ is partitioned into two disjoint graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, according to the following criterion. Vertices in G that belong to colour classes of size smaller (larger) than r in c, belong to V_1 (V_2). Colour classes of size exactly r can belong to either of the two vertex sets. Thus, V_1 and V_2 form a disjoint union of V. Let G_1 be the induced graph on V_1, and accordingly let G_2 be the induced graph on V_2.

Note that given the colouring c, its induced colouring c_1 on G_1 already admits, by definition, the resource limitation constraint, i.e., c_1 is an r-bounded vertex colouring of G_1. In contrast, the induced colouring on G_2, c_2, is not an r-bounded vertex colouring of G_2, and it therefore requires re-colouring. Given a colour class of size $p > r$ in c_2, we re-colour this colour class with $\frac{p}{r}$ distinct colours.[5] Since vertices that belong to a colour class form an independent set, such a re-colouring is always achievable. By repeating this re-colouring for all the colour classes in c_2 and by applying a disjoint union of all these re-colourings, we get an r-bounded vertex colouring $\tilde{c_2}$ of G_2. The disjoint union of c_1 and $\tilde{c_2}$, denoted $\tilde{c}$, is clearly an r-bounded vertex colouring of G.

It is interesting to investigate the conditions on the graph G and the parameter r under which the r-bounded vertex colouring $\tilde{c}$ that the above procedure yields is optimal (minimal). Such optimality can be ensured if the following three terms apply: (i) c_1 is a minimal r-bounded vertex colouring of G_1, (ii) $\tilde{c_2}$ is a minimal r-bounded vertex colouring of G_2, and (iii) the graph structure is such that in any optimal r-bounded vertex colouring, no two vertices $v_1 \in V_1$ and $v_2 \in V_2$ belong to the same colour class.

The induced colouring c_1 is an optimal colouring on G_1, otherwise we get a smaller colouring on G. Following our restriction to fractional colouring, it is obvious that $\tilde{c_2}$ is an optimal r-bounded vertex colouring of G_2, since it obtains the lower bound of $\frac{|V_2|}{r}$ colours. In this case, $\tilde{c_2}$ is actually an equitable $\frac{|V_2|}{r}$-colouring. If in addition the graph G_1 is *perfect* and all the vertices of G_2 are connected by edges to all the vertices of a maximum-size clique of G_1, then

[5] $\lceil \frac{p}{r} \rceil$ distinct colours should be used when non-fractional colouring is applied.

an optimal r-bounded colouring of G must colour V_1 and V_2 by disjoint sets of colours. These comprehensions lead to our next result regarding the case of a perfect graph with $r = 2$.

3.2 Achieving the mixed bound for $r = 2$

Given a perfect graph $G = (V, E)$ and the parameter $r = 2$, we will partition the graph into subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ that comply with those of the above r-bounded colouring procedure, but we will do so according to a specific construction process. As with the above colouring procedure, we start by colouring the graph G with $\chi(G)$ colours. However, out of all such possible optimal colourings, we choose the specific colouring c in which the number of vertices that belong to colour classes of size 1 or 2 is maximal.

According to the colouring procedure, the vertex set V_1 contains all the vertices that belong to colour classes of size 1 (which we hereinafter term *singleton*) in c, and may hold *some* of the vertices that belong to colour classes of size exactly $r = 2$. Accordingly, we start with V_1 consisting of only the vertices that belong to singleton colour classes in c, and present an inductive construction process in which V_1 gradually grows. Note that because the initial V_1 contains only singleton colour classes, it must form a clique, or else two or more of its vertices could share the same colour, and consequently contradict the optimality of the colouring c. During the inductive construction process this clique, termed Q ($Q \subseteq V_1$), will also gradually grow. $V_1(i)$ and $Q(i)$ represent the corresponding sets in step i of the induction. The inductive construction is as follows:

- **Base case:** $V_1(0) = \{v \in V | c(v) \text{ is singleton}\}$, $Q(0) = V_1(0)$.
- **Inductive step:** If, in step i, there exist two non adjacent vertices v and u such that $u \in Q(i-1)$, $v \notin V_1(i-1)$, and the colour class of $c(v)$ is of size 2, then let $V_1(i) = V_1(i-1) \cup \{v, w\}$, where w is the vertex that is coloured by the same colour as v, i.e., $c(w) = c(v)$. In addition, let $Q(i) = Q(i-1) \cup \{w\}$.

Note that in any step of the construction, $V_1(i)$ remains in accordance with V_1 of the r-bounded colouring procedure, since only vertices from colour classes of size exactly $r = 2$ are added to $V_1(i)$ at each step. Also, due to the fact that graph G is perfect, $Q(i)$ remains a clique after adding vertex w. This is because the addition of $\{v, w\}$ adds a new colour to $V_1(i)$, i.e., $\chi(G_1(i)) = \chi(G_1(i-1)) + 1$. In a perfect graph, the chromatic number of every induced subgraph equals the size of its largest clique. Thus, the increment in the chromatic number of G_1 in step i also increments the size of G_1's largest clique in step i, which corresponds to the addition of vertex w to the clique $Q(i)$.

The inductive construction process terminates at some step t when there do not exist two such vertices u and v that can lead to the construction of step $i = t + 1$. Let the final constructed sets be termed $V_1 = V_1(t)$ and $Q = Q(t)$.

Before we can utilize the discussed construction for proving that the mixed bound B_m can be attained, we need to prove the following lemma:

Lemma 1 *For any vertex $q \in Q$ there exists a 2-bounded optimal vertex colouring of V_1 such that q is a singleton.*

Proof We prove the lemma by induction on the steps i in which the respective sets $Q(i)$ and $V_1(i)$ are generated.

- **Base case:** In step $i = 0$, $Q(0) = V_1(0)$ consists of only singleton colour classes according to the original colouring c, which is optimal, so the lemma's assertion is correct.
- **Induction hypothesis:** For any vertex $q \in Q(i-1)$ there exists a 2-bounded optimal vertex colouring of $V_1(i-1)$ such that q is a singleton.
- **Inductive step:** According to the inductive construction of the clique Q, in each step $i > 0$, $Q(i) = Q(i-1) \cup \{w\}$, where $\{v, w\}$ forms a colour class of size 2, and vertex v is not adjacent to some vertex u in $Q(i-1)$. According to the induction hypothesis, the lemma's assertion is correct for every vertex $q \in Q(i-1)$, so what remains to prove is that there exists a 2-bounded optimal vertex colouring of $V_1(i)$ in which vertex w is a singleton. By the induction hypothesis, there is a 2-bounded optimal colouring of $V_1(i-1)$, say c', in which vertex u is a singleton. Now, it only remains to extend the colouring c' to $V_1(i)$ by setting the colour of vertex v to $c'(u)$, which in turn leaves vertex w with a new singleton colour in c'. Since $\{u, v\}$ forms a colour class of size 2 in c', we get a 2-bounded optimal vertex colouring of $V_1(i)$ in which vertex w is a singleton, as required. ∎

Now we turn to our main result.

Theorem 1 *The mixed bound $B_m = \max\{\chi(G_1) + \frac{|V_2|}{r}\}$ is tight for perfect graphs and $r = 2$.*

Proof Let $G = (V, E)$ be a perfect graph with $r = 2$, and let V_1 and $V_2 = V \setminus V_1$ be the vertex sets (and G_1 and G_2 their respective induced graphs) that resulted from the inductive construction process presented at the beginning of this subsection. We re-colour the vertices of V_2 in accordance with the r-bounded colouring procedure of Section 3.1. We claim that the fractional bounded chromatic number of G is $\chi_2^*(G) = \chi(G_1) + \frac{|V_2|}{2}$, thus proving the theorem. Following the characteristics of the r-bounded colouring procedure, it suffices to show that each vertex in the clique $Q \subseteq V_1$ is adjacent to all the vertices in V_2.

Consider two vertices $q \in Q$ and $v \in V_2$ (after termination of the inductive construction). According to Lemma 1, there exists a 2-bounded optimal vertex colouring c' of V_1 in which q is a singleton. Assume that v is not adjacent to q. In case vertex v belongs to a colour class of size 2, then the inductive construction would not have terminated, so we get a contradiction. In case vertex v belongs to a colour class of size larger than 2, then one can re-colour vertex v with $c'(q)$, which contradicts the maximality (in the number of vertices that belong to colour classes of size 1 or 2) of colouring c, which also holds for any optimal re-colouring of V_1, e.g., colouring c'. Thus, we conclude that every vertex $q \in Q$ is adjacent to every vertex $v \in V_2$, and so the B_m bound is attained.

Note that other valid partitions of graph G to G_1 and G_2 may lead to lower bounds than B_m, but these looser bounds would not be achievable. This is the reason for B_m being the maximal of all bounds of this form. ■

3.3 Counter example for $r = 3$

In the previous chapter, we have defined a 2-bounded colouring procedure for perfect graphs. At the end of the procedure, the graph vertices are partitioned into two sets: V_1 with a maximal clique Q, and V_2 whose vertices are all adjacent to all the vertices in Q. This partition defines a non-trivial mixed bound, larger than the clique bound and the resource bound, and this bound is achievable for $r = 2$. In contrast, as shown in the following example, for a 3-bounded colouring, the mixed bound is not always achievable.

Consider the conflict graph G shown in Figure 4(a). There is no non-trivial partition of the vertices leading to a mixed bound higher than the resource bound and the clique bound. So $B_m = B_c = B_r = 2$. As we shall prove in what follows, an optimal 3-bounded colouring of G is of size $2\frac{1}{3}$. Hence, the bounds are unachievable. For this example, the optimal 3-bounded colouring is obtained by the procedure given in Section 3.1. We start with a unique minimal colouring (up to isomorphism) of G, shown in Figure 4(b). Figure 4(c) shows the partition of the vertices into two sets: V_1, containing vertices of size less than $r = 3$, and V_2, with vertices of size higher than 3. In Figure 4(d) the vertices of V_2 are recoloured to obtain a bounded fractional colouring of size $2\frac{1}{3}$. In the remainder of the section, we shall prove that the colouring of Figure 4(d) is indeed optimal.

We use the following definitions and notations:

- Given a graph G we denote by $\mathcal{I}$ the set of all independent vertex sets of G.
- The set of all independent vertex sets that contain a vertex v is denoted $\mathcal{I}(v)$.
- A (regular) fractional colouring of G is a function $c : \mathcal{I} \to R+$ such that for each vertex v, $\sum_{I \in \mathcal{I}(v)} c(I) = 1$. Each colour is identified by an independent set I with $c(I) > 0$.
- Given a fractional colouring c, we denote by $Col_c(v)$ the set of colours that appear in v, i.e., $Col_c(v) = \{I | I \in \mathcal{I}(v), c(I) > 0\}$. For example, given the colouring of Figure 4(d), $Col_c(v_1) = \{\{v_1, v_3, v_5\}, \{v_1, v_3, v_6\}, \{v_1, v_5, v_6\}\}$.

To show that the conflict graph colouring given in Figure 4(d) is indeed optimal we will first show that there is an optimal fractional 3-bounded colouring for which both v_2 and v_4 are coloured the same, i.e., a colouring with $c(\{v_2, v4\}) = 1$.

Given the conflict graph of Figure 4(a), let c be an optimal 3-bounded colouring of G with $c(\{v_2, v4\})$ maximal. Suppose indirectly that $c(\{v_2, v4\}) < 1$. If so, there must be non-empty independent sets: $I \in Col_c(v_2) \setminus \{v_2, v_4\}$ and $J \in Col_c(v_4) \setminus \{v_2, v_4\}$. Without loss of generality suppose that $c(I) > c(J)$. Consider a new fractional colouring c' defined by $c'(\{v_2, v_4\}) = c(\{v_2, v_4\}) + c(J)$, $c'(J) = 0$, $c'(I) = c(I) - c(J)$, $c'((I \bigcup J) \setminus \{v_2, v_4\}) = c((I \bigcup J) \setminus \{v_2, v_4\}) + c(J)$, and $c'(X) = c(X)$ for any other independent set X. Clearly $c'(\{v_2, v_4\}) > c(\{v_2, v_4\})$ in contradiction to the maximality of $c(\{v_2, v4\})$.

Finally, since v_2 and v_4 are coloured the same, and each of the other four vertices are connected to either v_2 or v_4, these four vertices must be coloured by colours different from the colour of v_2 and v_4. So at least $\frac{4}{3}$ more colours are needed. Hence, the colouring of Figure 4(d) is optimal.

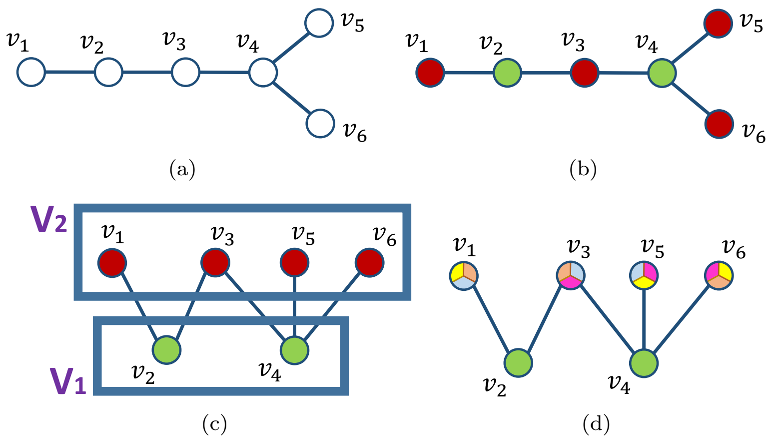

Fig. 4 (a) A conflict graph G with unachievable bounds for $r = 3$. (b) An optimal colouring of G. (c) Partitioning of the vertices into V_1 and V_2. (d) An optimal fractional colouring of G.

4 Discussion

In this paper, we have revisited the PCOSS problem – a natural generalization of the known open shop scheduling problem. We have previously studied two variants of PCOSS, with and without preemption. PCOSS without preemption is equivalent to the problem of conflict graph orientation [4,5]. Therefore, even for a single job, this is an NP-hard problem. Conversely, PCOSS with preemption is equivalent to a colouring problem, which may be easier to solve [7, 8]. For example, when the conflict graph is known to be perfect, the problem is polynomially solvable. In this paper, we have presented a variant of PCOSS with preemption and an additional limitation of r resources. We started with the issue of bounds. For an open shop scheduling problem with preemption two natural bounds exist, the clique bound and the resource bound – and the maximal bound among these is polynomially achievable [2,13]. However, for PCOSS these bounds may not always be achievable. Therefore, our first contribution is a new bound, which is a combination of the clique and resource bounds. Our second contribution is a procedure for r-bounded vertex colouring that combines regular colouring and equitable colouring. We proceeded to show that a specific implementation of this procedure yields an optimal fractional bounded colouring for perfect graphs and $r = 2$, which constitutes our third contribution. The bound is generally not achievable for $r = 3$.

The equivalence of the bounded colouring problem with $r = 2$ and the regular matching problem was mentioned by Hansen et al. [6] and Janssen and Kilakos [9]. It will be interesting to make the connection between the bound B_m for a 2-bounded colouring problem and the known bounds in the duality theorem for fractional matching [11]. It is interesting to study the relation between the polyhedral approach used to prove a MinMax theorem for the 2-bounded (fractional) chromatic number and our purely combinatorial approach.

Our starting point for the study of bounded colouring was the PCOSS model, and the timetabling problem of assigning technicians in a garage that inspired it. Limited resources, such as vehicle lifts, are common elements in many realistic problems, and having the ability to deal with them brings us one step closer to solving full-scale complex problems. In a previous work [8], we have presented a special variant of a PCOSS, termed uniform PCOSS, dealing with a problem where all of the jobs share the same conflicts. The conflict graph is then a Cartesian product between a one-job conflict graph (G_1) and a complete graph of size m (the number of machines). The main question is how the efficient solvability of the colouring of G_1 affects the hardness of the entire model. It is a matter for further research to ask analogous questions for the PCOSS with resources, i.e., bounded colouring.

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