

Polynomially solvable formulations for a class of nurse rostering problems

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Abstract Identifying underlying structures in combinatorial optimisation problems leads to a better understanding of a problem and, consequently, to efficient solution methodologies. The present paper introduces a new network flow formulation for a large class of nurse rostering problems. By solving an integer minimum cost flow problem in a carefully constructed network, nurses' shift schedules can be constructed in polynomial time. The performance of the new formulation is compared with a state of the art algorithm on a benchmark dataset. Computational experiments show that the new formulation performs better in terms of computation time, while still solving the problem to optimality. By identifying inherent combinatorial structures which can be efficiently exploited, insight is gained into the problem's complexity, thereby laying the foundations for a theory of nurse rostering.

Keywords Nurse rostering · Network flows · Mathematical programming

* Peter Brucker sadly passed away on July 24, 2013. His coauthors dedicate their contribution in this paper to his memory.

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1 Introduction

Scheduling nurses is a critical process in health care due to the high costs associated with these scarce resources. Nearly sixty years of research has been devoted to solving different variants of this problem, resulting in an equally large variety of solution techniques (Burke et al, 2004; Van den Bergh et al, 2013). Many nurse rostering problems addressed in the literature are complex in nature, dealing with a large variety of organisational and legal constraints (Brucker et al, 2010; Smet et al, 2013). Complex search algorithms have been proposed for dealing with such problems (Burke and Curtois, 2014; Valouxis et al, 2012). However, attention has also been paid to more straightforward, simplified variants of the problem which make abstraction of a large part of the operational complexity arising in practice. Studying the underlying structure of such problems can lead to valuable insights, resulting in improved methodologies for both simplified and complex nurse rostering problems. We revisit problems presented in the literature, and investigate whether they exhibit a combinatorial structure that can be efficiently exploited.

An example of such a combinatorial structure which has been given attention in personnel scheduling research is the use of network flow techniques (Ahuja et al, 1993). A common application of network flows is found in column generation approaches for personnel rostering, where the pricing problem is often modeled as a resource constrained shortest path problem (Jaumard et al, 1998). Networks have also been described to address more general problems: to calculate the size of a workforce (Koop, 1988), to reconstruct nurse rosters from a schedule with disruptions (Moz and Pato, 2004) or to allocate shift types to a fixed days-on roster (Dowland and Thompson, 2000). Brucker et al (2011) discuss networks for various (sub)problems related to personnel scheduling. Millar and Kiragu (1998) present a mathematical model with an underlying network structure to represent both a cyclic and non-cyclic nurse scheduling problem. Constraints regarding staffing demands and weekends are modelled as side constraints external to the network.

These network flow formulations make a strong abstraction of reality by e.g. assuming equal staffing requirements on all days or full staff availability, considering single shift scenarios and ignoring skill requirements. Furthermore, the number of contractual workforce constraints included in these formulations is typically limited to e.g. only restricting certain shift successions or only limiting the maximum number of consecutive assignments. The present paper fills the existing void by presenting a network flow model incorporating various practical and important nurse rostering constraints. Since there are no side constraints, the underlying structure of network flow problems is kept intact. Thereby, a solution methodology is established for efficiently solving a large class of nurse rostering problems.

The paper is organised as follows. Section 2 presents a detailed classification scheme for nurse rostering problems. Section 3 introduces a new network flow formulation for nurse rostering, along with several extensions. A computational

evaluation of the new formulation is presented in Section 4. Finally, Section 5 concludes the paper and identifies areas for future research.

2 A classification scheme for nurse rostering problems

De Causmaecker and Vanden Berghe (2011) present an $\alpha|\beta|\gamma$ classification scheme for practical nurse rostering problems (Table 1). The presented notation allows a wide variety of problem characteristics to be described. In this section, we introduce an extension to this classification scheme which allows detailed elements of nurse rostering problems to be described.

	Personnel constraints	Skill interactions	
α Personnel environment	A Availability	2, 3, ...	Fixed number
	S Sequences	N	Variable number
	B Balance	I	Individual skill definitions
	C Chaperoning		
β Work characteristics	Coverage constraints	Shift type	
	R Range	2, 3, ...	Fixed number
	T Time intervals	N	Variable number
	V Fluctuating	O	Overlapping
γ Optimisation objective	Objective	Mode	
	P Personnel constraints	M	Multi-objective
	L Coverage constraints		
	X Number of personnel		
	R Robustness		
	G General		

Table 1 Classification of nurse rostering problems (De Causmaecker and Vanden Berghe, 2011).

α : Personnel environment

The scheme of De Causmaecker and Vanden Berghe (2011) describes time-related (horizontal) constraints by $\alpha : A$ and $\alpha : S$ for counters and series, respectively. We present the following extensions to these constraint categories:

- $A \in \{a, \bar{a}, \underline{a}, \bar{s}, \underline{s}, \bar{s}, \bar{s}\}$ Type of counter constraint. When $A = a$ there is a constraint on the number of days worked, and when $A = s$ there is a constraint on the number of assignments of a particular shift type. The lines above and under each entry indicate the type of threshold. For example, a means that an exact number of days needs to be worked, \bar{a} means that only an upper bound is specified, \underline{a} refers to only a lower bound, finally, \bar{a} means that a range is defined.
- $S \in \{as, \bar{as}, \underline{as}, \bar{cs}, \underline{cs}, \bar{cs}, ss\}$ Type of series constraint. When $S = as$ there is a constraint on the number of consecutive days worked, when $S = ss$ there is a constraint on particular shift successions, and when $S = cs$ there is constraint on the number of consecutive assignments of a particular shift type. The threshold for this type of constraint is defined as in category $\alpha : A$.

β : Work characteristics

To detail the type of coverage constraint in a nurse rostering problem, the category $\beta : R$ is extended with the following elements:

- $R \in \{d, \bar{d}, \underline{d}, \bar{\underline{d}}\}$ Type of coverage constraint. The threshold for this type of constraint corresponds to the threshold definition in category $\alpha : A$.

γ : Optimisation objective

Several objectives can be described in the γ category. We present an extension to the category $\gamma : P$ to differentiate between different types of personnel related objectives:

- $P \in \{\sum wc, \sum px\}$ Objective function. $P = \sum wc$ denotes a weighted sum of soft constraint violations. When $P = \sum px$, the employee preferences are optimised.

While most common time-related constraints for nurse rostering are presented in the extended classification scheme, the use of the $\alpha|\beta|\gamma$ notation presents a flexible framework allowing for future extensions.

3 Network flow models for nurse rostering

3.1 Problem description

The scheduling period T is a set of t days $T = \{1, \dots, t\}$. There is a set S of s shift types $S = \{1, \dots, s\}$. On each day j and for each shift type k , arbitrary minimum and maximum staffing demands $0 \leq d_{jk}^l \leq d_{jk}^u$ are specified. The workforce N is a heterogeneous set of n nurses $N = \{1, \dots, n\}$. Each nurse i has to work exactly a_i days in T . Finally, each nurse i has a preference for working shift type k on day j , expressed as an inversely proportional integer cost c_{ijk} .

Let \mathcal{P} denote the problem of assigning shifts to nurses such that the staffing requirements are satisfied. Each nurse must work exactly the number of specified days and can be assigned to at most one shift per day. The objective is to minimise the costs c_{ijk} .

\mathcal{P} can be formulated as an integer linear program (ILP) with one set of decision variables

$$x_{ijk} = \begin{cases} 1 & \text{if nurse } i \text{ works shift } k \text{ on day } j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{P} : \min \sum_{i \in N} \sum_{j \in T} \sum_{k \in S} c_{ijk} x_{ijk} \quad (1)$$

$$s.t. \sum_{k \in S} x_{ijk} \leq 1 \quad \forall i \in N, j \in T \quad (2)$$

$$d_{jk}^l \leq \sum_{i \in N} x_{ijk} \leq d_{jk}^u \quad \forall j \in T, k \in S \quad (3)$$

$$\sum_{j \in T} \sum_{k \in S} x_{ijk} = a_i \quad \forall i \in N \quad (4)$$

$$x_{ijk} \in \{0, 1\} \quad \forall i \in N, j \in T, k \in S \quad (5)$$

The objective function 1 minimises the sum of costs incurred by the shift assignments. Constraints 2 ensure that at most one shift is assigned per day, per nurse. Constraints 3 model the minimum and maximum staffing demands. Constraints 4 restrict the number of days each nurse should work in the planning period. Finally, constraints 5 bound the decision variables.

Following the extended $\alpha|\beta|\gamma$ notation presented in Section 2, the class of problems we address is in $A(a)NI|R(\underline{d})VN|P(\sum px)$.

3.2 Network flow formulation

Problem \mathcal{P} can be reformulated as an integer minimum cost network flow problem in a directed network $G = (V, E)$, with V the set of nodes and E the set of arcs. The set V consists of four subsets of nodes.

Shift nodes For each day $j \in T$ and each shift type $k \in S$, a node is created representing the demand on day j for shift type k .

Time nodes For each nurse $i \in N$ and each day $j \in T$, a node is created representing a day on which a nurse can work.

Nurse nodes For each nurse $i \in N$, one node is created.

Other nodes There is one source node s and one sink node f .

Figure 1 shows the structure of the network G . Each shift node has one incoming arc from the source node. Its outgoing arcs are directed towards the time nodes corresponding to the day for which the shift node is defined. Each nurse node only has incoming arcs from time nodes associated with the nurse. Finally, each nurse node has one outgoing arc to the sink node.

Lemma 1 *The number of nodes in G is equal to $t(s + n) + n + 2$.*

Proof The network contains ts shift nodes, nt time nodes, n nurse nodes and two other nodes. \square

Lemma 2 *The number of arcs in G is equal to $t(s + n(s + 1)) + n$.*

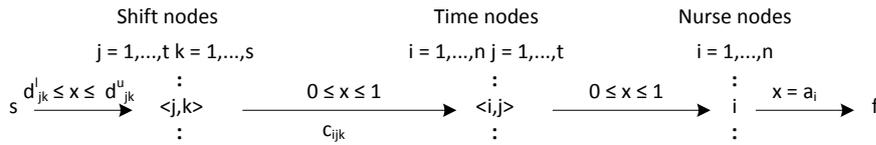


Fig. 1 Network G for problem \mathcal{P} . x denotes the flow through an arc.

Proof There are ts arcs going from the source node to the shift nodes. Each shift node has n arcs to time nodes. There are ts shift nodes, so in total tsn arcs go from shift nodes to time nodes. Each time node has one outgoing arc to a nurse node. With nt time nodes, nt arcs exist between the time nodes and the nurse nodes. Finally, there are n arcs between the nurse nodes and the sink node. \square

Flow costs are only defined on the arcs between the shift nodes and the time nodes, representing the cost c_{ijk} of assigning a nurse i to shift type k on day j . All nodes, except the source and sink nodes, are transshipment nodes. The supply in the source node is $\sum_{i \in N} a_i$, corresponding to the total number of days the nurses can work according to their contracts. The supply in the sink node is equal to $\sum_{i \in N} -a_i$.

Lower and upper bounds on the capacity of the arcs are appropriately defined to correctly represent problem \mathcal{P} . The arcs between the source node and the shift nodes have a lower (upper) bound equal to the minimum (maximum) staffing demand. Arcs between the nurse nodes and sink node have a lower and upper bound equal to the required number of days worked. All other arcs require a flow of either 0 or 1.

Theorem 1 *An optimal integer minimum cost flow in the network G corresponds to an optimal solution for problem \mathcal{P} .*

Proof Due to the construction of network G , a minimum cost solution respecting the capacity and demand constraints can be converted to a solution for problem \mathcal{P} . A flow on an arc between a time node defined for nurse i , day j and a nurse node defined for nurse i , corresponds to a working day for nurse i . By forcing a flow of a_i in the arc between the nurse node i and the sink node, nurse i will work exactly a_i days. Shift assignments are determined by flows in the arcs between the shift nodes and the time nodes. A flow from the shift node associated with day j , shift k to the time node associated with day j , nurse i , corresponds to nurse i working shift k on day j , thereby incurring cost c_{ijk} . The flow conservation constraints ensure that at least d_{jk}^l , and at most d_{jk}^u units of flow will be divided among the arcs leaving the associated shift node, thereby fulfilling the staffing demands. Since there is an upper bound of one on the arcs between the shift nodes and time nodes, a nurse cannot be assigned more than one shift per day. \square

3.3 Extensions

Several elements can be added to the definition of problem \mathcal{P} , which can also be included in the network formulation.

3.3.1 Unavailabilities

A *shift unavailability* prevents the assignment of shift type k on day j . A set of shift unavailabilities $\bar{S}_{ij} \subseteq S$ can be defined, containing the shifts for which nurse i is unavailable on day j . This is enforced by adding constraints (6) in the ILP model.

$$\sum_{k \in \bar{S}_{ij}} x_{ijk} = 0, \forall i \in N, j \in T \quad (6)$$

Shift unavailabilities can be modeled in network G by setting the capacity upper bound to zero on the arcs going from the shift node associated with each shift in \bar{S}_{ij} to the corresponding time nodes.

This type of unavailability can also be used to include qualification requirements for particular shifts. For example, when one head nurse is required during the day shift, a dedicated *head nurse-day shift* can be created. The capacity upper bound on the arcs going from the associated shift nodes should be zero, except for the arcs to the time nodes defined for the actual head nurses.

A *day unavailability* forbids the assignment of any shift on day j . Again, for each nurse i , a set of day unavailabilities $\bar{T}_i \subseteq T$ can be defined. Constraints (7) model these unavailabilities in the ILP model.

$$\sum_{k \in S} x_{ijk} = 0, \forall i \in N, j \in \bar{T}_i \quad (7)$$

In the network G , day unavailabilities are enforced by changing the capacity upper bound to zero on the arcs going from the relevant time nodes to the corresponding nurse nodes.

3.3.2 Hard preferences

Hard preferences, either for working days or for particular shifts, can be modeled in a similar way as the unavailabilities. An assignment of shift k on day j for nurse i can be fixed by adding constraint (8) to the ILP model.

$$x_{ijk} = 1 \quad (8)$$

For a fixed day-on assignment on day j to nurse i , constraint (9) should be added to the ILP model.

$$\sum_{k \in S} x_{ijk} = 1 \quad (9)$$

In network G , instead of setting the capacity upper bound on selected arcs to zero, the capacity lower bound is set to one, thereby forcing a flow through the arcs and consequently ensuring a working day or shift.

3.3.3 Daily employment cost

There exist cases in which a cost c_i is incurred for each day nurse i works in the planning period. The objective function in the ILP includes an additional term to represent these costs (expression (10)).

$$\sum_{i \in N} \sum_{j \in T} \sum_{k \in S} c_{ijk} x_{ijk} + \sum_{i \in N} \sum_{j \in T} c_i \sum_{k \in S} x_{ijk} \quad (10)$$

This extension can be modeled in network G by adding a flow cost equal to c_i on the arcs from the nurse nodes to the sink node. Since each unit of flow through these arcs represents one day of labour, a flow cost corresponds to the cost c_i .

3.3.4 Ranged constraint

Problem \mathcal{P} requires nurse i to work exactly a_i days. This constraint can be relaxed such that nurse i works between a_i^l and a_i^u days. In the ILP, constraints (4) are replaced by constraints (11).

$$a_i^l \leq \sum_{j \in T} \sum_{k \in S} x_{ijk} \leq a_i^u, \forall i \in N \quad (11)$$

This relaxation is included in the network flow model by transforming the network G to a circulation network G' by adding one arc from the sink node to the source node. There is no cost associated with this arc, and the capacity is only bounded below by zero. All nodes become transshipment nodes. According to Theorem 1, an integer minimum flow in G' again corresponds to an optimal solution for problem \mathcal{P} with constraints (11).

3.3.5 Weighted constraint

Consider the modification of problem \mathcal{P} such that for each nurse $i \in N$ only an upper bound a_i on the number of days worked is imposed, which can be violated at the cost of a penalty w_i per additional day worked. This is modelled by replacing constraints (4) with constraints (12) in the ILP.

$$\sum_{j \in T} \sum_{k \in S} x_{ijk} \leq a_i + p_i, \forall i \in N \quad (12)$$

The variable p_i represents the number of days nurse i works over the allowed maximum. Violations of this constraint are minimised by optimising objective function (13).

$$\sum_{i \in N} \sum_{j \in T} \sum_{k \in S} c_{ijk} x_{ijk} + \sum_{i \in N} w_i p_i \quad (13)$$

In the network flow formulation, one additional transshipment node is created for each nurse: a constraint node. Each of these new nodes is connected with the nurse node of the corresponding nurse, and the sink node. Both arcs have positive infinite capacity. By adding a flow cost equal to w_i on the arcs between the nurse nodes and the constraint nodes, the penalty for additional days worked is counted. Figure 2 shows the modified network.

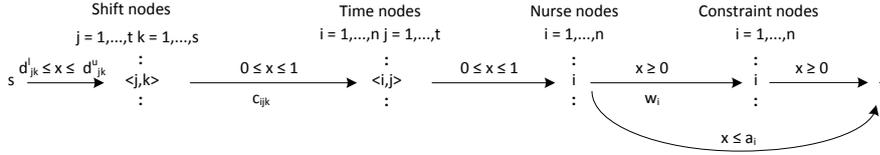


Fig. 2 Network G for problem \mathcal{P} with weighted constraint violation. x denotes the flow through an arc.

To construct a solution for this extended problem description, the network G depicted in Figure 2 is first transformed to a circulation network G' , such that all nodes become transshipment nodes. According to Theorem 1, an integer minimum flow in G' again corresponds to an optimal solution for problem \mathcal{P} with the weighted constraint.

4 Computational analysis

4.1 Applying the network flow model

The effectiveness of the new network flow formulation is evaluated by analysing a series of computational experiments on the NSPLib benchmark dataset (Vanhoucke and Maenhout, 2007). The dataset consists of different constraint sets that can be combined with any of the 29,160 problem instances. By omitting the constraint on forbidden shift sequences from the NPSLib problem description, a subset of instances (case 1 constraint set for the 7-day instances, and case 9 for the 28-day instances) corresponds to problem \mathcal{P} .

As in problem \mathcal{P} , an assignment cost c_{ijk} is defined for each nurse, shift, day combination. Since NSPLib uses a dummy shift k' to represent a day-off, costs of working shifts are scaled relative to the cost of the dummy shift. Let $c_{ijk'}$ be the preference cost of nurse i for a day-off on day j . To correctly incorporate the use of the dummy shift in the network G , the original costs c_{ijk} are transformed to modified costs c'_{ijk} by applying equation 14.

$$c'_{ijk} = c_{ijk} - c_{ijk'} \quad \forall i \in N, j \in T, k \in S \quad (14)$$

It is clear that a problem instance with costs c'_{ijk} has the same optimal assignments as a problem with costs c_{ijk} since the relative differences in preference remain the same. The only difference is that c'_{ijk} can be less than zero, which in general does not influence (optimal) choices made by algorithms. Since after applying equation 14, the costs associated with days-off are zero, their assignment can be omitted from any shift assignment model. The costs associated with the other shifts will determine whether a particular shift type assignment is preferable to a day-off.

4.2 Performance evaluation

We first present a comparison in terms of size for both the ILP formulation and the network flow formulation (NF). Table 2 shows, for the different instances in NSPLib, the number of variables and constraints in the ILP model and the number of nodes and arcs in the network G .

Days	Nurses	ILP		NF	
		Variables	Constraints	Nodes	Arcs
7	25	700	228	230	928
	50	1400	428	430	1828
	75	2100	628	630	2728
	100	2800	828	830	3628
28	30	3360	982	984	4342
	60	6720	1852	1854	8572

Table 2 Size comparison of ILP and network flow models.

We performed a series of experiments with the ILP formulation and network flow formulation to solve instances from NSPLib. The experiments were carried out on an Intel Core i5 CPU at 2.5GHz with 4GB RAM operating on Windows 7, using a single thread. All algorithms were coded in C++. IBM ILOG CPLEX 12.5 was used to solve the ILP formulation. The network flow formulation was solved with the network simplex algorithm in LEMON 1.3.

Table 3 compares the solution costs and computation times in seconds for the ILP formulation (ILP) and the network flow formulation (NF). These values are averages over all instances, grouped per number of days and number of nurses.

Both the ILP and network flow formulations obtain optimal solutions for all instances, while requiring very little computation time. The reported calculation times are plotted in Figure 3 as functions of problem size, determined by the number of days and nurses. For both approaches, the trend shows that an increasing problem size, and thus an increasing number of variables, constraints or network dimensions, leads to longer calculation times. However, for the network flow formulation, the required calculation time is up to a magnitude lower than for the ILP formulation, thereby demonstrating the ad-

Days	Nurses	ILP		NF	
		Avg. cost	Time(s)	Avg. cost	Time(s)
7	25	245.41	0.0206	245.41	0.0014
	50	489.77	0.0324	489.77	0.0031
	75	740.11	0.0447	740.11	0.0062
	100	1191.19	0.0579	1191.19	0.0102
28	30	1422.32	0.0685	1422.32	0.0134
	60	2915.64	0.1406	2915.64	0.0406

Table 3 Comparison of the ILP and network flow formulations.

vantage of exploiting the problem’s underlying combinatorial structure with well known efficient algorithms.

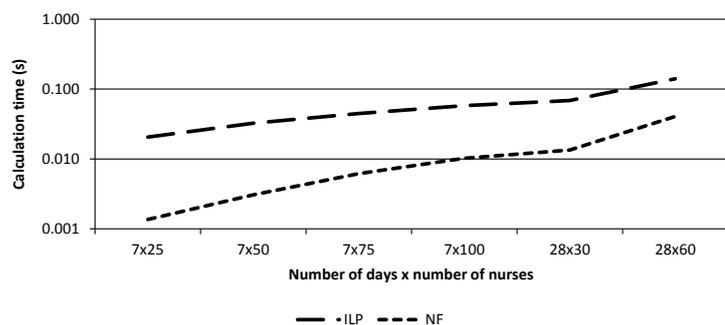


Fig. 3 Required calculation times in function of problem size.

5 Conclusions and future work

By reformulating nurse rostering problems as integer minimum cost flow problems, we identified a class of problems in $A(a)NI|R(\bar{d})VN|P(\sum px)$ that can be solved in polynomial time. Within this problem class, several variants are introduced which can be modeled by making minor modifications to the presented flow network, while still preserving its combinatorial structure. The contribution lies in this new formulation by which a large class of problems can be solved in polynomial time.

Computational experiments demonstrated the effectiveness of the new formulation on a benchmark dataset from the literature. Compared to solving an ILP formulation with a state of the art mathematical solver, a network simplex algorithm required almost ten times less computation time for solving the integer minimum cost flow problem in the presented flow network.

The challenge of identifying efficiently exploitable combinatorial structures for more complex problems, incorporating other practical constraints, remains

an important research area. Such results lead to establishing a theory of personnel scheduling which is severely lacking in the academic community. Understanding various problems' structure and complexity supports the study of more complex nurse rostering problems arising in practice.

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